Brian Tomasik
Transfinite Numbers – Requirement 4, Spring 2005

Logicism was founded around the year 1884 by the German philosopher and mathematician Gottlob Frege, and it was further developed by Peano, Bertrand Russell, and Alfred North Whitehead. The school of thought held as its basic premise that all of mathematics follows naturally from logic. Russell and Whitehead defined "logic" in their 1910 *Principia Mathematica* as follows: "A logical proposition is a proposition which has complete generality and is true in virtue of its form rather than its content." For example, the "law of the excluded middle"—which states that either *p* or *~p* must be true—would be considered a logical proposition because it originates from the "form" of language itself, not from the "content" of any specific examples. However, the "axiom of infinity"—which holds that there exist infinite sets—is not a logical proposition inasmuch as it arises not from logical necessity but only from the many examples of infinity that humans encounter in the real world. Neither the axiom of infinity nor the axiom of choice—two of the nine axioms of the modern Zermelo and Fraenkel (ZF) set theory—can be considered logical propositions according to the Russell and Whitehead definition; thus, from this perspective, logicism has failed to ground contemporary mathematics.

The mathematical school of logicism has its roots in the philosophical doctrine of realism, which in medieval scholarship meant the Platonic belief that truths exist outside of the human mind waiting to be discovered. Accordingly, abstract mathematical ideas can be taken to exist without demonstrating how the human mind would "construct" them. I disagree with this philosophy and thence with the concept of logicism to which it leads. To me, truth is whatever the mind decides to make it, and there are uncountably infinite different ways to perceive reality—most of which humans cannot even imagine. While intuitive logical truths usually work for practical purposes, humans have no reason to assert that they must be absolute or that another system of logic could not work just as well under an alternate perspective of reality. And even within the framework of the prevailing human view of reality, intuitive conclusions have often turned out to be wrong, as advances in the theory of relativity, quantum physics, and countless other fields have shown.

The school of intuitionism—founded around 1908 by the Dutch mathematician L. E. J. Brouwer—took an entirely different course from logicism. While the logicists tried to justify the existing conclusions of classical mathematics, the intuitionists rejected them as flawed and full of contradictions (citing, for example, the paradoxes that Russell had found in Cantor's set theory). Instead, they sought to rebuild the entire system of mathematics from the beginning.

Intuitionism originates from the idea of natural numbers; its adherents believe that people have an automatic, "intuitive" sense of the number one, and in repeating that thought process, they arrive at the number two, the number three, and any other positive integer. Thus, rather than simply proving that a natural number exists, intuitionists "construct" it. Intuitionists do not accept logical statements until they can be demonstrated constructively (that is, through mathematics). Thus, the intuitionists adopt a diametric position from the logicists: while the latter attempt to prove mathematics through logic, the former prove logic through mathematics. Even though intuitionism, unlike logicism, contains no internal contradictions, it has largely been repugned by the mathematical community, mainly because it has not succeeded in reproducing all of the conclusions of classical mathematics and has on occasion produced entirely different results, such as the theorem that every function is continuous which has real values and is defined for all real numbers.

Intuitionism mirrors closely the philosophical doctrine of conceptualism, which holds that abstractions exist only to the extent that the mind has constructed them. As I stated before, I personally subscribe to this philosophy, and hence, I agree most with intuitionistic mathematics. Instead of deducing from a set of assumed logical absolutes, I would much prefer to demonstrate a conclusion in the almost inductive manner of the intuitionists, for the latter method is much less likely to involve errors or create contradictions.

Unlike logicists and intuitionists, formalists deal not with abstract entities but with language. Though formalist ideas had been expressed during the late 1800s, the true foundation of formalism was not until around 1910 by German mathematician David Hilbert. The school of thought centers around "formalization"—the process of expressing an axiomatized theory in terms of a first-order language (that is, a language consisting of mathematical symbols). The purpose of formalization is to demonstrate the truth of a given theory using the "Hilbert program," by which one shows that, working entirely within the formalized language of a given theory, one cannot arrive at a contradiction.

The philosophy best connected with formalism is that of nominalism, which holds that abstract entities do not exist—whether in the human mind or outside of it—except as words and names. In a similar way, formalist mathematicians concern themselves not with the abstractions found within a given theory but merely with the language used to represent the theory.

Like logicism, formalism has run into problems, most notably Kurt Gödel's demonstration in 1931 that "No sentence of [the formalized language] *L* which can be interpreted as asserting that [the given theory] *T* is free of contradictions can be proven formally within the language *L*." In other words, the Hilbert program does not work. Thus, of the three schools of thought on the foundation of mathematics discussed above, only intuitionism has so far been flawless. For this reason, and because the philosophy of conceptualism is closest to my own, I sympathize most strongly with the school of intuitionism.