Derivation of the Formula
for the Derivative of any Monomial

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This formula for the derivative of any monomial will presently be proved:

$$\text{If } y = x^n\text{, then } (dy)/(dx) = nx^{n-1}$$

Here is the equation from which this derivation begins:

$$\text{If } y = f(x)\text{, then } (dy)/(dx) = [f(x + h) - f(x)]/h,$$
$$\text{in the limit that } h \rightarrow 0$$

Before proceeding onward, it will be useful to review binomial expansion. As an illustration, consider the following example:

$$(x + h)^3$$
$$= (x + h)(x + h)(x + h)$$
$$= (x^2 + xh + xh + h^2)(x + h)$$
$$= x^3 + x^2h + x^2h + xh^2 + x^2h + xh^2 + xh^2 + h^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$

Notice the pattern of the exponents of x and h. Notice, too, the coefficients of the terms when arranged in descending order for x: 1 3 3 1. Each coefficient is the result of a combination, ${}_nC_r$, in which n equals the power to which (x + h) is raised (namely, 3) and r equals the number of the term minus one. Thus, the expression could be written in this way:

$$({}_3C_0)x^3 + ({}_3C_1)x^2h + ({}_3C_2)xh^2 + ({}_3C_3)h^3$$

In general, then, the expansion of $(x + h)^n$ can be written thus, assuming n to be a natural number:

$$({}_nC_0)x^nh^0 + ({}_nC_1)x^{n-1}h^1 + ({}_nC_2)x^{n-2}h^2 + \ldots + ({}_nC_{n-1})x^1h^{n-1} + ({}_nC_n)x^0h^n$$

Returning to the derivative formula, if $f(x) = x^n$, then (dy)/(dx) will equal $[(x + h)^n - x^n]/h$, in the limit that $h \rightarrow 0$. Substituting the expansion of $(x + h)^n$ produces

$$(dy)/(dx) =$$

$$([({}_nC_0)x^nh^0 + ({}_nC_1)x^{n-1}h^1 + ({}_nC_2)x^{n-2}h^2 + \dots + ({}_nC_{n-1})x^1h^{n-1} + ({}_nC_n)x^0h^n] - x^n)/h,$$
in the limit that $h \rightarrow 0$

It is always true that ${}_nC_0 = 1$ and that $h^0 = 1$. Wherefore, the first term in the expansion of $(x + h)^n$ becomes simply x^n, in which form it cancels with the $-x^n$ to yield

$$(dy)/(dx) = [({}_nC_1)x^{n-1}h^1 + ({}_nC_2)x^{n-2}h^2 + \dots + ({}_nC_{n-1})x^1h^{n-1} + ({}_nC_n)x^0h^n]/h,$$
in the limit that $h \rightarrow 0$

Each term in the numerator now contains at least one h, making possible division by the denominator:

$$(dy)/(dx) = ({}_nC_1)x^{n-1} + ({}_nC_2)x^{n-2}h^1 + \dots + ({}_nC_{n-1})x^1h^{n-2} + ({}_nC_n)x^0h^{n-1},$$
in the limit that $h \rightarrow 0$

Because h is approaching zero, all terms containing h as a factor become zero:

$$(dy)/(dx) = ({}_nC_1)x^{n-1} + 0 + \dots + 0 + 0$$

There are always n ways in which to take one element from a set containing n elements. Hence, ${}_nC_1$ will always equal n. And, assuming n to be a natural number,

$$(dy)/(dx) = nx^{n-1}$$

works this for fractional and negative values of n??? what if n = 1 or something? if negatives and fractions work not, how do I prove that n must be a positive integer so that my end result contains the same restrictions on the definition of n as the thing I set out to prove???