Derivation of the Formula for the Derivative of any Monomial
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This formula for the derivative of any monomial will presently be proved:

If \( y = x^n \), then \( \frac{dy}{dx} = nx^{n-1} \)

Here is the equation from which this derivation begins:

If \( y = f(x) \), then \( \frac{dy}{dx} = \frac{f(x + h) - f(x)}{h} \)

in the limit that \( h \to 0 \)

Before proceeding onward, it will be useful to review binomial expansion. As an illustration, consider the following example:

\[
(x + h)^3 = (x + h)(x + h)(x + h) = (x^2 + xh + xh + h^2)(x + h) = x^3 + x^2h + x^2h + xh^2 + xh^2 + h^3 = x^3 + 3x^2h + 3xh^2 + h^3
\]

Notice the pattern of the exponents of \( x \) and \( h \). Notice, too, the coefficients of the terms when arranged in descending order for \( x \): 1 3 3 1. Each coefficient is the result of a combination, \( \binom{n}{r} \), in which \( n \) equals the power to which \( (x + h) \) is raised (namely, 3) and \( r \) equals the number of the term minus one. Thus, the expression could be written in this way:

\[
(\binom{3}{0})x^3 + (\binom{3}{1})x^2h + (\binom{3}{2})xh^2 + (\binom{3}{3})h^3
\]

In general, then, the expansion of \( (x + h)^n \) can be written thus, assuming \( n \) to be a natural number:

\[
(\binom{n}{0})x^0h^n + (\binom{n}{1})x^1h^{n-1} + (\binom{n}{2})x^2h^{n-2} + \ldots + (\binom{n}{n-1})x^{n-1}h + (\binom{n}{n})x^0h^n
\]

Returning to the derivative formula, if \( f(x) = x^n \), then \( \frac{dy}{dx} \) will equal \( \frac{[(x + h)^n - x^n]}{h} \), in the limit that \( h \to 0 \). Substituting the expansion of \( (x + h)^n \) produces

\[
\frac{dy}{dx} =
\]
\[ \left( (\binom{n}{0})x^n h^0 + (\binom{n}{1})x^{n-1} h^1 + (\binom{n}{2})x^{n-2} h^2 + \ldots + (\binom{n}{n-1})x^1 h^{n-1} + (\binom{n}{n})x^0 h^n \right) - x^n \right) / h, \]
in the limit that \( h \to 0 \)

It is always true that \( \binom{n}{0} = 1 \) and that \( h^0 = 1 \). Wherefore, the first term in the expansion of \( (x + h)^n \) becomes simply \( x^n \), in which form it cancels with the \( -x^n \) to yield

\[ \frac{dy}{dx} = \left[ (\binom{n}{1})x^{n-1} h^1 + (\binom{n}{2})x^{n-2} h^2 + \ldots + (\binom{n}{n-1})x^1 h^{n-1} + (\binom{n}{n})x^0 h^n \right] / h, \]
in the limit that \( h \to 0 \)

Each term in the numerator now contains at least one \( h \), making possible division by the denominator:

\[ \frac{dy}{dx} = (\binom{n}{1})x^{n-1} + (\binom{n}{2})x^{n-2} h^1 + \ldots + (\binom{n}{n-1})x^1 h^{n-2} + (\binom{n}{n})x^0 h^{n-1}, \]
in the limit that \( h \to 0 \)

Because \( h \) is approaching zero, all terms containing \( h \) as a factor become zero:

\[ \frac{dy}{dx} = (\binom{n}{1})x^{n-1} + 0 + \ldots + 0 + 0 \]

There are always \( n \) ways in which to take one element from a set containing \( n \) elements. Hence, \( \binom{n}{1} \) will always equal \( n \). And, assuming \( n \) to be a natural number,

\[ \frac{dy}{dx} = nx^{n-1} \]

works this for fractional and negative values of \( n \)???: what if \( n = 1 \) or something? if negatives and fractions work not, how do I prove that \( n \) must be a positive integer so that my end result contains the same restrictions on the definition of \( n \) as the thing I set out to prove???